Optimal Harvesting Strategies for Stochastic Single-Species, Multiage Class Models*

ROY MENDELSSOHN
Southwest Fisheries Center,
National Marine Fisheries Service, NOAA,
Honolulu, Hawaii 96812

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ABSTRACT

Qualitative properties of optimal harvesting policies for stochastic, single-species, multiage class models are described. For many problems the k-dimensional problem (k is the number of age classes) can be reduced to k one-dimensional problems, which can be solved far more readily. When such separability does not occur, bounds can be put on the derivatives of an optimal policy function which can greatly increase computational efficiency.

1. INTRODUCTION

In a series of recent papers, I have explored optimal policy functions for stochastic single-species, pooled-age-class models under a variety of assumptions on the one-period return. The basic results are given in [14]; these have been extended to include "smoothing costs" due to changes in the sizes of the harvest [12] and also to include multiple objectives [13]. The purpose of this paper is to extend this analysis to single-species, multiage class models.

The age distribution of the population affects harvesting strategies due to age-dependent mortality, reproduction, and size. For a given total population size x, the growth over the next years will vary greatly if the population is mainly in the younger rather than the older age classes. Mortality will vary. Reproduction due both to the first age of reproduction and to age-dependent reproductive rates will vary. The average weight of the catch will vary, since older animals tend to be larger also. Computer models that have been used to study the effects of management policies on sandhill

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cranes [15], elephants [7], and deer [1] show that age structure may have effects that are obscured by predictions based only on the total population size and the initial management policy.

A population model which includes the age distribution of the population raises conceptual problems with such traditional goals of resource management as maximum sustained yield. For example, a particular model of the population dynamics, for a given initial population distribution, may not permit any policy that ensures constant yield, constant total number harvested, or constant age distribution. Even if there exists a policy that realizes any or all of these goals, it may require a long span of time to bring the initial population into this equilibrium configuration, and the result may well be either under- or overexploitation of the resource during the nonequilibrium period. The problems encountered in trying to define a maximum sustained yield harvest for a Leslie matrix model of population growth is discussed by Mendelssohn [11]. This includes differing definitions given by Williamson [22], Watt [21], Usher [18], Dunkel [6], Doubleday [5], and Rorres and Fair [17]. The underlying dilemma in each of these articles is that of trying to extend to higher dimensions concepts that only have meaning in one dimension.

The models in this paper contain only limited biological detail as an initial examination of stochastic harvesting models with age structure. Deterministic models presented by Goh [8] and Clark et al. [3], to be described later in the paper, include age structure with very restrictive assumptions. Less restrictive models are presented in this paper, and they suggest possibilities and pitfalls for future research. For example, one might think that the results of Mendelssohn and Sobel [14] should extend into k-dimensions. Unfortunately, they do not. Some of the results of that paper rest on the fact that the real line is completely ordered by the inequality $\leqslant$. In other words, if x^1 is not less than or equal to x, then x^1 must be greater than x. However, coordinate-by-coordinate comparisons of k-vectors induce only partial orderings. For two k-vectors, $\mathbf{x}^1$ and $\mathbf{x}^2$, the falsity of $\mathbf{x}^1 \leqslant \mathbf{x}^2$ does not imply $\mathbf{x}^1 \geqslant \mathbf{x}^2$. Therefore, in higher dimensions, most of the proofs in [14] are invalid. As a result, the analyses in this paper are more complex, the assumptions are stronger, and the results weaker than those in that paper. Nevertheless, interesting results are obtained by using the same basic approach and methods as in that paper on two interesting classes of models. The first model assumes random recruitment and random age-dependent survivorship *rates*, both independent of population density. This model is a generalization of the classic Beverton-Holt model [2]. The second model differs from the first by assuming that recruitment is a random concave function of the total population size, analogous to the Ricker equation [16]. The first model has the mathematical property of "separability" or "additivity." As a result, it will be possible to find an optimal policy for each cohort independently of the rest of the population. A variety of weaker results are presented for the second model. The strongest result assumes that there is a linear return p^i from each age class, and that the sequence $\{p^i\}$, $i=1,\ldots,k$, is concave. With these assumptions, it is proven that an optimal policy has a "new better than used" (NBU) property, that is, the oldest are always harvested first.

Both models assume complete selectivity in harvesting the age classes. In practice, harvesting techniques lie between the two extremes of complete selectivity and random sampling. The assumption of complete selectivity allows a determination of the target if such control is possible, and lends insight into the crucial factors of the population dynamics if the best of all harvests is possible. One extension to include gear selectivity leads to "pulse" harvesting of the resource.

The notation here is the k-dimensional extension of that in [14]. Let the ith component x_t^i of the k-vector $\mathbf{x}_t=(x_t^i)$ denote the number alive in the ith age class at the beginning of period t. Similarly $\mathbf{y}_t$ $(\mathbf{z}_t)$ is the vector of the number of individuals left (harvested) in the k age classes at the end of (during) period t. $\mathbf{d}_t$ is now a random vector whose components may be jointly distributed. The sequence $\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_T$ is assumed to consist of independent and identically distributed random vectors distributed as the generic random vector $\mathbf{d}$. $A_t(\mathbf{x})$ denotes a k-vector-valued function, a policy function, with component functions $a_t^i(\mathbf{x})$, and $S[\mathbf{y},\mathbf{d}]$ is a k-vector valued (transition) function with components $s^i[\mathbf{y},\mathbf{d}]$. The one period return function $G(\mathbf{x},\mathbf{y})$ maps a subset of $\mathbb{R}^{2k}$ into $\mathbb{R}^1$. And finally, instead of (left) derivatives, the (left) gradient is evaluated. The gradient of a function f is written ∇f, and its ith component $\nabla^i f$. If f is a function mapping a subset of $\mathbb{R}^{2k}\to\mathbb{R}^1$, then $f^{[1]}$ is the gradient with respect to the first vector argument, and $f^{[2]}$ is the gradient with respect to the second vector argument. Thus, with obvious changes in interpretation, Eq. (4.1) in [14] remains the generic problem, that is:

$$\begin{aligned} &f_0(\cdot)\equiv 0,\\ &f_n(\mathbf{x})=\max_{0\leq \mathbf{y}\leq \mathbf{x}}\{J_n(\mathbf{x},\mathbf{y})\}, \end{aligned} \tag{1}$$

where

$$J_n(\mathbf{x},\mathbf{y})=G(\mathbf{x},\mathbf{y})+\alpha E\{f_{n-1}(S[\mathbf{y},\mathbf{d}])\}.$$

2. MODELS AND RESULTS

A model basic to fisheries research assumes constant recruitment each year, and survivorship functions which depend only on the size of each age class. Mathematically, this becomes

$$\begin{aligned} &x_{t+1}^1=R,\\ &x_{t+1}^{i+1}=s^i[y_t^i], \qquad i=1,\ldots,k-1. \end{aligned}$$

A special case of this model is the Beverton-Holt model [2], which has been discussed in the context of dynamic optimization by Goh [8] and Clark et al. [3].

Examples of fish populations that behave in a manner consistent with the Beverton-Holt model can be found in the references cited above or in [9] or [16]. However, it is rare that recruitment is truly constant, or that survivorship functions fit the data exactly; rather there is usually "noise" around this expected value. In the case of recruitment, this can be modeled by assuming that the constant (mean) level is modified by a random variable, that is,

$$x_{t+1}^1 = d^1 R, \qquad \infty > d^1 \geqslant 0$$

or

$$x_{t+1}^1 = R - d^1, \qquad 0 \leqslant d^1 \leqslant R.$$

In what follows, the multiplicative form d^1R will be assumed. There are two reasons for this. First, as has been discussed elsewhere [16, 20], if there are a large number of environmental effects acting on the number of recruits, then the random variable will tend to act multiplicatively and have a log normal distribution. Second, all the proofs presented for the multiplicative case are true for the additive case, with the obvious necessary modifications. Thus, it is without loss of generality that only the multiplicative case is examined here. The complete stochastic model can be described as

$$\begin{aligned} x_{t+1}^1 &= s^1[\mathbf{y},\mathbf{d}] = d^1 R, \quad d^1R \geqslant 0 \qquad \text{for all } d^1,\\ x_{t+1}^{i+1} &= s^{i+1}[\mathbf{y},\mathbf{d}] = s^{i+1}[y^i, d^{i+1}], \qquad i = 1,\ldots,k-1. \end{aligned} \tag{2}$$

For the return from harvest, assume that the benefit is proportional to the weight of each class age, i.e., it is given by $\sum_{i=1}^k p^i(x^i - y^i)$, $p^i \geqslant 0$ for all i. If the cohorts are separated in space, as is often the case for ocean fish that spawn upriver, costs for harvesting each cohort should also be independent except for a per diem operating cost which is independent of whether harvesting is done. Let $g^i(x^i, y^i)$ be the cost of harvesting the ith age class from x^i to y^i individuals, and let $c\sum_{i=1}^k (x^i - y^i)$ be the total operating cost (that is, the cost is proportional to the time spent harvesting, which in turn is proportional to the total number harvested). Then the return function is

$$G(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{k} \{(p^i - c)(x^i - y^i) - g^i(x^i, y^i)\}. \tag{3}$$

The next theorem states that with the assumption of Eqs. (2) and (3), an optimal policy can be determined for each cohort independently. This is a useful result, because a k-dimensional problem has been reduced to k one-dimensional problems, where the stronger results of [14] apply.

THEOREM 1

In the dynamic program (1), *if* $S[\mathbf{y},\mathbf{d}]$ *is assumed to be given by Eq.* (2) *and* $G(\mathbf{x},\mathbf{y})$ *is assumed to be given by Eq.* (3), *then:*

(i) *the problem is divisible into k subprograms that consider each cohort independently;*

(ii) *for each* $n, t \geqslant n \geqslant k$, $a_n^i(\mathbf{x}) = a_n^i(x^i) = a^i(x^i)$ *for all* i *and* $\mathbf{x} \in X$, *where* $a_n^i(x^i)$ *denotes that* a_n^i *depends on* $\mathbf{x}$ *only through* x^i.

Proof. Part (i) is essentially proven in [19]. Part (ii) follows because for $n \geqslant k$, each cohort is independent of all other cohorts and $S[\mathbf{y},\mathbf{d}]$, $G(\mathbf{x},\mathbf{y})$ are stationary functions. ■

Theorem 1 does not describe an optimal policy but rather proves that the problem of finding an optimal policy can be greatly simplified. To see the usefulness of Theorem 1, consider the Beverton-Holt model as a special case of Eq. (2):

$$\begin{aligned} &x_{t+1}^i = d^1R, \qquad d^1R \geqslant 0, \\ &x_{t+1}^{i+1} = d^{i+1}s^i y_t^i, \quad 0 \leqslant d^{i+1}s^i \leqslant 1, \qquad i = 1,\ldots,k-1. \end{aligned} \tag{4}$$

The model assumes that given $d^{i+1}s^i$ a fixed proportion of the cohort survives to the next period, but that this proportion is "noisy", i.e., it varies at random from year to year. Note that the model does not meet the assumptions of two similar stochastic processes: first, where s^i is assumed to be a Bernoulli random variable, or second, where each individual, when born, has a known lifetime distribution. Assume that the return function is given by

$$G(\mathbf{x},\mathbf{y}) = \sum_{i=1}^{k} p^i(x^i - y^i), \qquad p^i \geqslant 0 \quad \text{for all } i. \tag{5}$$

THEOREM 2

For a dynamic programming problem given by Eq. (1), *if* $S[\mathbf{y},\mathbf{d}]$ *is given by Eq.* (4) *and* $G(\mathbf{x},\mathbf{y})$ *by Eq.* (5), *then:*

(i) *for each cohort born with n years remaining in the planning horizon, which can obtain a maximum age* $k(n)$ *within the planning horizon, there* *exists an age $j^*(n)$, $1 \leqslant j^*(n) \leqslant k(n)$, such that an optimal policy for the cohort is as follows:*

Age	*Action*
$1 \leqslant i \leqslant j^*(n)$	*Do not harvest.*
$j^*(n)$	*Harvest entire remaining cohort.*

(ii) *For $n \geqslant k, j^*(n) \equiv j$.*

Proof. From Theorem 1, the dynamic programming problem is separable by cohort. For each cohort, there is a dynamic program of $k(n)$ periods given by

$$\begin{gathered} f_0 \equiv 0, \\ f_i(x) = \max_{0 \leqslant y \leqslant x} \{ p^{i} \cdot (x^i - y^i) + \alpha E f_{i+1}(d^{i+1} s y^i) \}, \\ i = 1, \ldots, k(n). \end{gathered} \tag{6}$$

The dynamic program (6) is separable by individual. The problem for each individual becomes

$$\begin{aligned} &f_0 \equiv 0, \\ &f_i(x) = \max\{ p^i, \alpha E f_{i+1}(d^{i+1} s^i) \}, \qquad i = 1, \ldots, k(n), \end{aligned} \tag{7}$$

and the decision is either to harvest this period or not.

Given an individual just born Eq. (7) becomes

$$\begin{aligned} f_1 = \max\{ &p^1, E(d_n^2) \alpha p^2 s^1, E(d_n^2 d_{n-1}^3) \alpha^2 p^3 s^1 s^2, \ldots, \\ &E(d_n^2 d_{n-1}^3 \ldots d_{n-k(n)-1}^{k(n)}) \alpha^{k(n)-1} p^{k(n)} s^1 s^2 \ldots s^{k(n)-1} \}. \end{aligned} \tag{8}$$

Let $j^*(n)$ be the age where f_1 achieves its maximum; that is, $j^*(n)$ is the age where an individual of the cohort reaches a maximum value. From Eq. (8),

$$\begin{aligned} &E(d_{n-1}^{i+1} \ldots d_{n-1-j^*(n)}^{j^*}) \alpha^{j^*(n)-i-1} p^{j^*}(n) s^i \ldots s^{j^*(n)-1} \\ &\qquad \geqslant E(d_{n-1}^{i+1} \ldots d^j) \alpha^{j-i-1} p^j s^i \ldots s^{j-1} \end{aligned} \tag{9}$$

for all i, $1 \leqslant i \leqslant j^*(n)$, and all j, $1 \leqslant j \leqslant k(n)$. From Eq. (7) for $i \leqslant j^*(n)$,

$$\begin{aligned} f_i = \max\{ &p^i, E(d_{n-1}^{i+1}) \alpha p^{i+1} s^i, \ldots, \\ &E(d_{n-1}^{i+1} d_{n-i-1}^{i+2} \ldots d_{n-k(n)-1}^{k(n)}) \alpha^{k(n)-i-1} p^{k(n)} s^i s^{i+1} \ldots s^{k(n)-1} \}. \end{aligned}$$

However, Eq. (9) implies that f_i achieves its maximum at age $j^*(n)$, and the proof of part (i) is complete.

Part (ii) follows immediately from Theorem 1, part (ii) and the result from part (i) of this theorem. ∎

The policy described by Theorem 2 harvests a single age class each year. Since weight, length, and other measurements of size are usually well correlated with age, it is the type of target policy that might well be achieved in practice.

Goh [8] examines the Beverton-Holt model as a continuous-time, deterministic problem, and assumes that $s^i = s$ for all age classes and that $\{p^i\}$ is nondecreasing in i. Clark et al. [3] also assume that $s^i = s$ for all i, that $\{(p^{i+1} - p^i)/p^i\}$ is a nondecreasing sequence, and that harvesting removes each age class in proportion to its relative number in the population. Clark et al. use harvesting effort as their decision variable, and prove that "pulse"-type fishing is optimal, that is, in each period fishing effort should be either at a zero level or at a maximum level. The next corollary proves that if there is a function of effort which can be treated as a decision variable, and if the catch in each age class is proportional to this variable, then for $0 \leqslant e \leqslant e_{\max}$ (a constrained decision), an optimal policy chooses an effort of zero or $e_{\max}$.

COROLLARY

In the dynamic program (1) *let* $S[\mathbf{y}, \mathbf{d}]$ *and* $G(\mathbf{x}, \mathbf{y})$ *be given by Eqs.* (4) *and* (5), *respectively. Assume that in each period the only decision possible is total fishing effort* e *such that* $0 \leqslant e \leqslant e_{\max}$, *and that for a given* e *and an initial population vector* $\mathbf{x}$, *the catch from each age class in period* t *is* $eq_t^i x_t^i$. *Then an optimal policy in each period chooses either zero effort or else* $e_{\max}$.

Proof. The corollary will be proven by showing that the resulting dynamic programming problem is convex (linear) in the decision e. It then follows from [10] that the maximum must occur at a boundary point, that is, either $e_{\max}$ or zero. At period $n = 1$, the return is

$$e \sum_i p^i q_1^i x_1^i.$$

On the set $\{e : 0 \leqslant e \leqslant e_{\max}\}$ the return is linear and nondecreasing in e for all $\mathbf{x} \in X$, so that $e_{\max}$ is optimal in period 1, for all values of $\mathbf{x}$. At $n = 2$, for a given $\mathbf{x}$ and e, the return is:

$$e \sum_i p^i q_2^i x_2^i + \alpha E \left\{ e_{\max} \left[p^1 q_1^1 R + \sum_{i=1}^{k-1} p^{i+1} d^{i+1} s^i q_1^i \cdot (x_2^i - e q_2^i x_2^i) \right] \right\}$$

Again, the return is convex (linear) in e, so either $e_{\max}$ or zero is optimal. Assume the theorem is true in periods $2, 3, \ldots, n-1$. At period n, for given $\mathbf{x}$

and e, the return is

$$e\sum_i p^i q^i x^i + \alpha E f_{n-1}(S[\mathbf{x},e,\mathbf{d}]).$$

However, by the induction hypothesis and Eq. (4), $\alpha E f_{n-1}(S[\mathbf{x},e,\mathbf{d}]$ is convex in e, and the proof is complete. ■

In the fisheries literature the q_t^i's are called the catchability coefficients. The corollary relaxes the proof of Clark et al. [3] by allowing for age-dependent and time-dependent catchability. The crux of the proof is that the decision variable in each period is constrained neither by the state in that period nor by the decision in the period before.

Several authors (see the references in Sec. 1) have suggested that a Leslie matrix can model an exploited population that is at a level where resource limitations have no significant effect. Mendelssohn [10] demonstrates that an optimal policy for this model, assuming returns are linear, is either to harvest the entire population immediately or else to let the population grow without bound. The Beverton-Holt model assumes constant recruitment; the Leslie matrix assumes additive, proportional recruitment of the form $\Sigma F^i y^i$. A reasonable model might lie somewhere between these two models. As one possibility, consider recruitment given by a random, concave function, similar perhaps to the Ricker equation, and survivorship rates given by a random vector $\mathbf{ds}$ as in Eq. (3). Formally, the transition function becomes:

$$x_{t-1}^1 = s^1\left[\sum_{i=1}^k y^i, d_t^1\right],$$

$$x_{t+1}^{i+1} = d_t^{i+1} s^i y_t^i, \quad 0 \leqslant d^{i+1} s^i \leqslant 1, \qquad i=1,\dots,k-1, \tag{10}$$

where $s^1[\cdot, d^1]$ is concave for each value of $\mathbf{d}$.

The next theorem examines harvesting strategies for the population model described in Eq. (10), with a linear return function as in Eq. (5). Part (a) of the theorem assumes $s^1[\cdot, d^1]$ is nondecreasing for each fixed $\mathbf{d}$. Consider an individual age i in some period. Then $\alpha^{j-i-1} p^j E(d^{i+1} d^{i+2} \dots d^j) s^i s^{i+1} \dots s^{j-1}$ is the expected discounted return from that individual at age j, without considering reproduction. If the expected discounted return at age j is greater than p^i, there is an optimal policy such that no individuals are harvested from that age class in that period.

Part (b) of the theorem assumes $d^{i+1} s^i = ds$ for all age classes i, but $s^1[\cdot, d^1]$ need not be nondecreasing. Then it is shown that for each period n and for all initial populations $\mathbf{x} \in X$, the total number remaining in the population satisfies

$$0 \leqslant \sum_{i=1}^k a^i(\mathbf{x}^1) - \sum_{i=1}^k a^i(\mathbf{x}) \leqslant \sum_{i=1}^k x^{1i} - \sum_{i=1}^k x^i \qquad \text{if} \quad \mathbf{x}^1 \geqslant \mathbf{x}.$$

Part (c) is similar to part (b), but it is assumed also that the p^i's form a concave sequence. This would be true if the growth curve were a concave function of time, as in the von Bertalanffy growth equation. With these assumptions, an optimal harvesting strategy has an NBU property, that is, an older individual is always harvested before a younger one.

THEOREM 3

In the dynamic program (1) *assume* $S[\mathbf{y},\mathbf{d}]$ *is given by Eq.* (10) *and* $G(\mathbf{x},\mathbf{y})$ *is given by Eq.* (5).

(a) *Suppose* $s^1[\cdot,\mathbf{d}]$ *is nondecreasing, continuous, and concave for each* $\mathbf{d}$, *and let* $k^i(n)$ *be the largest age obtainable by an individual of age* i *in period* n. *Suppose that for some* j *width*

$$i < j \leqslant k^i(n),$$
$$\alpha^{j-i-1} E\left(d_n^{i+1} d_{n-1}^{i+2} \dots d_{n-j-1}^{j}\right) p^j s^i s^{i+1} \dots s^{j-1} \geqslant p^i. \tag{11}$$

Then no individuals are harvested from age class i *in period* n.

(b) *If* $s^1[\cdot,\mathbf{d}^1]$ *is concave and continuous, and if* $d^{i+1}s^i = ds$ *for all* i, $i=1,\dots,k-1$, *then for each* n *and all* $\mathbf{x}\in X$,

$$0 \leqslant \sum_{i=1}^{k} \alpha^i(x^1) - \sum_{i=1}^{k} \alpha^i(x) \leqslant \sum_{i=1}^{i} x^{1i} - \sum_{i=1}^{k} x^i \qquad \text{if} \quad x^1 \geqslant x.$$

(c) *If, under the assumption of part* (b), *it is assumed that the sequence* $\{p^i\}$ *is concave in* i, *then an NBU policy is optimal.*

Proof. Essential to the proof of this theorem is that the theorem in [14] readily extends to k dimensions. This implies that $J_n(\mathbf{x},\mathbf{y})$ is concave and continuous, and that $f_n(\mathbf{x})$ is concave, continuous, and nondecreasing. Also, consider

$$\text{Maximize} \quad E \sum_{t=1}^{T} \sum_{i=1}^{k} \alpha p^i (x_t^i - y_t^i) \tag{12}$$

subject to $0 \leqslant \mathbf{y}_t \leqslant \mathbf{x}_t$ and Eq. (10). Substituting $S[\mathbf{y}_{t-1},\mathbf{d}]$ for $\mathbf{x}_t$, $t=2,\dots,T$, Eq. (12) becomes

$$\text{Maximize} \quad E\left\{ \sum_{t=1}^{T} \sum_{i=1}^{k} \alpha^{t-1} p^i \left(s^{i-1}[\mathbf{y}_t,\mathbf{d}_t] - y_t^i\right) + \sum_{i=1}^{k} p^i x_1^i - \alpha^{T-1} \sum_{i=1}^{k} p^i y_T^i \right\}. \tag{13}$$

The term $\sum_{i=1}^{k} x_1^i$ is independent of the decision process, and Eq. (13) clearly obtains a maximum at $\mathbf{y}_T \equiv 0$. Let $G(\mathbf{y})$ be given by

$$G(\mathbf{y}) = \sum_{i=1}^{k} p^i(\alpha E(s^{i-1}[\mathbf{y},\mathbf{d}] - y^i)).$$

Then the dynamic programming problem becomes

$$f_0(\cdot) \equiv 0,$$
$$f_n(x) = \max_{\mathbf{0} \leqslant \mathbf{y} \leqslant \mathbf{x}} \{G(\mathbf{y}) + \alpha E f_{n-1}(S[\mathbf{y},\mathbf{d}])\}.$$

Let $J_n(\mathbf{y})$ be defined as

$$G(\mathbf{y}) + \alpha E f_{n-1}(S[\mathbf{y},\mathbf{d}]).$$

Part (a). At period $n = 1$,

$$\nabla^i J_1(\mathbf{y}) = \alpha E(d^{i+1}) p^{i+1} s^i - p^i + \alpha p^1 E s^{[1]}\Big[\sum y^i, d^1\Big].$$

Assume, as in the theorem, that $\alpha E(d^{i+1}) p^{i+1} s^i - p^i$ is nonnegative. By assumption, $s^{[1]}[\sum y^i, d^1]$ is nonnegative, so that $\nabla^i J_1(\mathbf{y})$ is nonnegative for all $\mathbf{y}$. From the necessary and sufficient conditions for an optimum [10], this implies that $a_1^i(\cdot) \equiv x^i$.

Assume the theorem is true in periods $1, 2, 3, \ldots, n-1$. At period n,

$$\nabla^i J_n(\mathbf{y}) = \alpha E(d^{i+1}) p^{i+1} s^i - p^i + \text{terms nonnegative for all } \mathbf{y}. \tag{14}$$

If $\alpha E(d^{i+1}) p^{i+1} s^i - p^i$ is nonnegative, the theorem is immediate. If not, and there exists an age j such that Eq. (11) is true, then it follows that

$$\alpha^{j-i-1} E(d^{i+1} d^{i+2} \ldots d^j) p^j s^i s^{i+1} \ldots s^{j-1} \geqslant p^i > \alpha E(d^{i+1}) p^{i+1} s^i. \tag{15}$$

From the inductive hypothesis, this equation implies that no individuals are harvested from age class $i+1$ in period $n-1$. Therefore, Eq. (14) becomes

$$\nabla^i J_n(\mathbf{y}) = \alpha^2 E(d^{i+1} d^{i+2}) p^{i+2} s^i s^{i+1} - p^i$$
$$+ \text{terms nonnegative for all } \mathbf{y}.$$

Again, either $\alpha^2 E(d^{i+1} d^{i+2}) p^{i+2} s^i s^{i+1} - p^i$ is nonnegative, or else the same argument can be repeated up to age j, where the expression is nonnegative by assumption. Thus $\nabla^i J_n$ is nonnegative for all y, and this implies that $a_n^i(\cdot) \equiv x^i$ for all $\mathbf{x} \in X$.

Part (b). Consider the transformation of variables

$$
\begin{aligned}
w_t^1 &= x_t^1, & v_t^1 &= y_t^1,\\
w_t^2 &= x_t^1 + x_t^2, & v_t^2 &= y_t^2 + y_t^1,\\
&\vdots & &\vdots\\
w_t^k &= \sum_{i=1}^{k} x_t^i, & v_t^k &= \sum_{i=1}^{k} y_t^i.
\end{aligned}
$$

Then the dynamic programming problem becomes

$$
\begin{aligned}
&f_0(\cdot) \equiv 0,\\
&f_n(\mathbf{w}) = \max_{\substack{w^{i+1}-w^i \geqslant v^{i+1}-v^i \geqslant 0\\ i=1,\dots,k-1}} \{J_n(\mathbf{v})\}
\end{aligned}
$$

where

$$J_n(\mathbf{v}) = \sum_{i=1}^{k} c^i v^i + \alpha E f_{n-1}(S[\mathbf{v},\mathbf{d}])$$

and

$$c^i = (\alpha\mu s p^{i+2} - p^{i+1}) - (\alpha\mu s p^{i+1} - p^i), \qquad \mu \equiv E(d)$$

$S[\mathbf{v},\mathbf{d}]$ is given by

$$
\begin{aligned}
&s^1[\mathbf{v},\mathbf{d}] = s^1[v^k, d^1]\\
&s^j[\mathbf{v},\mathbf{d}] = s^1[v^k, d^1] + ds \sum_{i=1} (v^{i+1} - v^i), \qquad j = 2,\dots,k-1.
\end{aligned}
$$

Suppose $v^k, v^{k-1},\dots,v^2$ are fixed. Then there is an optimal v^1 given by $v^{1*}(\mathbf{w}, v^k, v^{k-1},\dots,x^2)$. Now suppose $v^k,\dots,v^3$ are fixed. Then there is an optimal v^2 depending only on $\mathbf{w}$ and $v^k, v^{k-1},\dots,v^3$. This can be repeated until the problem is reduced to choosing an optimal v^k given $\mathbf{w}$, and then unraveling the optimal $v^{k-1}, v^{k-2},\dots,v^1$. Noting that $\mathbf{w}$ affects the return only through the constraint set, it is tedious but straightforward to prove for $\mathbf{w}^1 \geqslant \mathbf{w}$, with $(w^{1i+1} - w^{1i}) \geqslant (w^{i+1} - w^i)$ for all i, that $v_n^k(\mathbf{w}^1) \geqslant v_n^k(\mathbf{w})$. This implies

$$\sum_{i=1}^{k} a_n^i(\mathbf{x}) \leqslant \sum_{i=1}^{k} a_n^i(\mathbf{x}^1) \qquad \text{if} \quad \mathbf{x}^1 \geqslant \mathbf{x}. \tag{16a}$$

A similar transformation of variables (summing over the age classes) can be done when $\mathbf{z} = \mathbf{x} - \mathbf{y}$ is the decision variable, and a parallel argument yields

$$\sum_{i=1}^{k}\left[x^{i}-a_{n}^{i}(\mathbf{x})\right] \leqslant \sum_{i=1}^{k}\left[x^{1i}-a_{n}^{i}(\mathbf{x}^{1})\right] \qquad \text{if} \quad \mathbf{x}^{1} \geqslant \mathbf{x}. \tag{16b}$$

Together, the two inequalities (16a) and (16b) yield:

$$0 \leqslant \sum_{i=1}^{k} a_{n}^{i}(\mathbf{x}^{1})-\sum_{i=1}^{k} a_{n}^{i}(\mathbf{x}) \leqslant \sum_{i=1}^{k} x^{1i}-\sum_{i=1}^{k} x^{i} \qquad \text{if} \quad \mathbf{x}^{1} \geqslant \mathbf{x}.$$

Part (c). By assumption $\{p^{i}\}$ is concave in i, which implies that the sequence $\{p^{i+1}-p^{i}\}$ is a nonincreasing sequence. For some constant s, $1 \geqslant s \geqslant 0$,

$$\begin{aligned}(p^{i+1}-p^{i}) \geqslant (p^{i+2}-p^{i+1}) \quad &\Rightarrow \quad (p^{i+1}-p^{i}) \geqslant s(p^{i+2}-p^{i+1}) \\ &\Rightarrow \quad (sp^{i+1}-p^{i}) \geqslant (sp^{i+2}-p^{i+1}).\end{aligned} \tag{17}$$

Part (c) will be proven by showing that for each n and all $\mathbf{y}$, $\nabla^{i}J_{n}(\mathbf{y}) \geqslant \nabla^{j}J_{n}(\mathbf{y})$ if $i<j$. If the inequality is true for all $\mathbf{y}$, then it follows that an individual will be harvested from age class j before any are harvested from age class i, which is the desired NBU policy.

At period 1,

$$\nabla^{i}J_{1}(\mathbf{y})=\alpha E(d)p^{i+1}s-p^{i}+\alpha p^{i}Es^{[1]1}\left[\sum y^{i}, d^{1}\right]$$

and

$$\nabla^{i}J_{1}(\mathbf{y})-\nabla^{j}J_{1}(\mathbf{y})=\alpha E(d)s(p^{i+1}-p^{j+1})-(p^{i}-p^{j}). \tag{18}$$

Equation (18) is of the form $(sp^{i+1}-p^{i})-(sp^{j+1}-p^{j})$, $1 \geqslant s \geqslant 0$. For $i<j$, Eq. (17) implies that this must be nonnegative, so that (18) is nonnegative for all $i<j$.

Assume the theorem is true at periods $1,2,3,\ldots,n-1$. At period n,

$$\begin{aligned}\nabla^{i}J_{n}(\mathbf{y})-\nabla^{j}J_{n}(\mathbf{y})=&\left\{\left[\alpha E(d)sp^{i+1}-p^{i}\right]-\left[\alpha E(d)sp^{j+1}-p^{j}\right]\right\} \\ &+\alpha sE\left\{d\left[\nabla^{i+1}f_{n-1}(S[\mathbf{y},\mathbf{d}])-\nabla^{j+1}f_{n-1}(S[\mathbf{y},\mathbf{d}])\right]\right\}.\end{aligned}$$

Equation (17) implies that the first term on the RHS of Eq. (19) is nonnegative. An argument similar to that in [14] implies

$$\nabla^{i+1}f_{n-1}(\mathbf{x})=\left[\nabla^{i+1}J_{n-1}(A_{n-1}(\mathbf{x})\right]^{+}. \tag{20}$$

By the inductive hypothesis, $\nabla^{i+1}J_{n-1} \geqslant \nabla^{j+1}J_{n-1}$ for all $\mathbf{y}$ if $i \leqslant j$. This implies that $\nabla^{i+1}f_{n-1}(\mathbf{x}) \geqslant \nabla^{j+1}f_{n-1}(\mathbf{x})$ for all $\mathbf{x} \in X$ and that Eq. (19) is nonnegative if $i<j$. As noted at the start of the proof, the ordering of the gradient by age is sufficient to prove that an NBU policy is optimal. ∎

The next theorem proves that if $s^1[\cdot,d^1]$ is nondecreasing, then the assumptions of Theorem 3, part (c) imply $A_{n+1}(\mathbf{x}) \geqslant A_n(\mathbf{x})$.

THEOREM 4

The assumptions of Theorem 3, part (c) *and the additional assumption that* $s^1[\cdot,d^1]$ *is nondecreasing imply that there is an optimal policy such that for each n and all* $\mathbf{x}\in X$,

(i) $A_{n+1}(\mathbf{x}) \geqslant A_n(\mathbf{x})$
(ii) *as* $n\to\infty$, $A_n(\mathbf{x})\to A(\mathbf{x})$

Proof. $\nabla[f_1(\mathbf{x})-f_0(\mathbf{x})]=\nabla f_1(\mathbf{x})\geqslant \mathbf{0}$. Assume as an inductive hypothesis that $\nabla[f_n(\mathbf{x})-f_{n-1}(\mathbf{x})]\geqslant\mathbf{0}$. Then

$$\nabla J_{n+1}(\mathbf{y})-\nabla J_n(\mathbf{y})=\alpha E\left\{[\nabla f_n(S[\mathbf{y},\mathbf{d}])-\nabla f_{n-1}(S[\mathbf{y},\mathbf{d}])]\xi_d(\mathbf{y})\right\}\geqslant\mathbf{0}. \tag{21}$$

where $\xi_d(\mathbf{y})$ is the Jacobian matrix of $S[\mathbf{y},\mathbf{d}]$ for fixed $\mathbf{d}$. Equation (21) implies that $\nabla J_{n+1}(A_n(\mathbf{x}))\geqslant\nabla J_n(A_n(\mathbf{x}))$. Since an NBU policy is an optimal policy, if $A_n(\mathbf{x})$ is to be altered in order to obtain an optimum in period $n+1$, it must be altered by increasing the oldest nonzero age class. This implies that $A_{n+1}(\mathbf{x})\geqslant A_n(\mathbf{x})$.

To complete the induction, it is necessary to prove that $\nabla f_{n+1}(\mathbf{x})-\nabla f_n(\mathbf{x})\geqslant\mathbf{0}$. From the definition of $f_n(\mathbf{x})$,

$$\nabla^i f_{n+1}(\mathbf{x})-\nabla^i f_n(\mathbf{x})=\left\{\left[\nabla^i J_{n+1}A_{n+1}(\mathbf{x})\right]^+-\left[\nabla^i J_n A_n(\mathbf{x})\right]^+\right\}.$$

If $a_n^i(\mathbf{x})\neq x^i$, then $\nabla^i f_n(\mathbf{x})=0$, and it is trivially true that $\nabla^i f_{n+1}(\mathbf{x})\geqslant\nabla^i f_n(\mathbf{x})$. If $a_n^i(\mathbf{x})=x^i$, then $a_n^i(\mathbf{x})=a_{n+1}^i(\mathbf{x})=x^i$. It is possible that in this instance, $\nabla^i f_{n+1}(\mathbf{x})\leqslant\nabla^i f_n(\mathbf{x})$. However, for the induction, what is important is that $a_{n+1}^i(\mathbf{x})=x^i$ if $a_n^i(\mathbf{x})=x^i$, that is, that $\nabla^i f_{n+1}(\mathbf{x})$ be positive. *This* is implied by the induction, and therefore it is without loss of generality that $\nabla^i f_{n+1}(\mathbf{x})\geqslant\nabla^i f_n(\mathbf{x})$. This completes the proof of (i). The proof of (ii) is an immediate consequence of (i). Since $\{A_n(\mathbf{x})\}$ is a monotone nondecreasing sequence in n which has $\mathbf{x}$ as uniform upper bound, as $n\to\infty$ there exists an $A(\mathbf{x})$ such that $A_n(\mathbf{x})\to A(\mathbf{x})$. ∎

As in [14], the results of Theorems 3 and 4 can be used to increase computational efficiency as well as to lend insight into optimal harvesting strategies. The assumptions given in Theorem 3, part (c) yield the most complete results, so that model will be used to illustrate how the results improve computational efficiency.

There are two sets of results that apply. First, for each n and for all $\mathbf{x} \in X$,

$$0 \leqslant \sum_{i=1}^{k} a_n^i(\mathbf{x}^1) - \sum_{i=1}^{k} a_n^i(\mathbf{x}) \leqslant \sum_{i=1}^{k} x^{1i} - \sum_{i=1}^{k} x^i \quad \text{if} \quad \mathbf{x}^1 \geqslant \mathbf{x}.$$

Also, for each n, $A_n(\cdot)$ has an NBU structure. For each n, $A_n(\mathbf{0}) \equiv \mathbf{0}$ starts the algorithm. Given $A_n(\mathbf{x})$, consider $A_n(\mathbf{x}+\boldsymbol{\delta}^j)$, where $\boldsymbol{\delta}^j$ is a k-vector with zero in every component but the jth, which has a value of one. Let j^* be the last nonzero age class of $A_n(\mathbf{x})$. Then if $j \geqslant j^*$, $A_n(\mathbf{x}+\boldsymbol{\delta}^j) \equiv A_n(\mathbf{x})$. If $j < j^*$, then $a_n^i(\mathbf{x}+\boldsymbol{\delta}^j) = a_n^i(\mathbf{x})$ for $i \neq j, j^*$; $a_n^j(\mathbf{x}+\boldsymbol{\delta}^j) = a_n^j(\mathbf{x})+1$, and $a_n^{j^*}(\mathbf{x}+\boldsymbol{\delta}^j)$ is either $a_n^{j^*}(\mathbf{x})$ or $a_n^{j^*}(\mathbf{x})-1$. Thus, given $A_n(\mathbf{x})$, either $A_n(\mathbf{x}+\boldsymbol{\delta}^j)$ is completely determined, or else only two states, $A_n(\mathbf{x})+\boldsymbol{\delta}^j$ and $A_n(\mathbf{x})+\boldsymbol{\delta}^j-\boldsymbol{\delta}^{j^*}$, need to be evaluated.

In the infinite horizon problem, the convergence of $A_n(\mathbf{x})$ to $A(\mathbf{x})$ may allow $A(\mathbf{x})$ to be calculated by linear programming (see [4] for details). However, it should be noted that the convergence of $A_n(\mathbf{x})$ to $A(\mathbf{x})$ does not necessarily imply that $f_n(\mathbf{x})$ converges to an $f(\mathbf{x})$ for which $A(\mathbf{x})$ is an optimal stationary policy.

3. DISCUSSION AND COMMENTS

The results presented here generalize to specific multiage class models the results of [14]. The interpretations of the results presented here are close enough to the discussion in that paper that it will not be repeated here.

There are several results that would seem to be true, but so far have escaped proof. They are of interest both as areas of future research and also as a means of examining more closely the models discussed in this paper.

Consider the most general form of the model described in Theorem 3, that is,

$$x_{t+1}^1 = s^1\left[\sum y_t^i, d^1\right],$$
$$x_{t+1}^{i+1} = d^{i+1} s^i y_t^i.$$

It would seem reasonable that for $n \geqslant k$, the elements of the gradients of $J_n(\mathbf{y})$ should remain in a constant ordering. For $n < k$, the ordering changes because the oldest obtainable age class varies, and therefore the future value of the age class varies. For $n > k$, consider $\nabla^i J_n(\mathbf{y})$:

$$\begin{aligned} \nabla^i J_n(\mathbf{y}) = {} & \alpha E(d^{i+1}) s^i p^{i+1} - p^i \\ & + \alpha E\left\{\nabla^1 f_{n-1}(S[\mathbf{y},\mathbf{d}]) s^{[1]1}\left[\sum y^i, d^1\right]\right\} \\ & + \alpha s^i E\left\{d^{i+1} \nabla^{i+1} f_{n-1}(S[\mathbf{y},\mathbf{d}])\right\}. \end{aligned}$$

Only the last term will vary as $\mathbf{y}$ or n changes for any two age classes. The last term is the expected future change in value due to future reproduction and future growth. That the ordering of the change in future growth should remain invariant after k years has an intuitive appeal in terms of the population dynamics.

For the same problem, it would seem to be true that

$$0 \leqslant \frac{\partial a_n^i(\mathbf{x})}{\partial x^i} \leqslant 1, \tag{22}$$

$$-1 \leqslant \frac{\partial a_n^i(\mathbf{x})}{\partial x^j} < 0 \qquad \text{for all} \quad (i,j), \; i \neq j.$$

Equation (22) has been proven for the special case of $d^{i+1}s^i = ds$ for all $i = 1,\ldots,k-1$. For the more general model, it is possible to prove that $0 \leqslant \partial a_n^i(\mathbf{x})/\partial x^i \leqslant 1$, and also that $0 \geqslant \partial a_n^i(\mathbf{x})/\partial x^j$ for $i \neq j$. However, the lower bound has not been proven. The motivation for the lower bound is that individuals in each age class are substitutes for one another. The future discounted growth in weight of an individual is density-independent. Reproduction only depends on the total number in the population. An optimal policy should substitute a less valuable individual (in the future) in the harvest for a more valuable one, if the additional individual is available. The reason why the substitution should be at most 1:1 is that reproductive return is being balanced by the substitution, and this only depends on the total number remaining. While this argument may seem intuitively correct, no proof or counterexample has been found yet.

If $s^1[\cdot, d^1]$ is nondecreasing, it would seem that for all $n \geqslant k$, $A_{n+1}(\mathbf{x}) \leqslant A_n(\mathbf{x})$ for all $\mathbf{x} \in X$. Again, the argument hinges on the idea that after k periods the "end effects" due to a finite planning horizon disappear. The proof would be similar in spirit to Theorem 3, but it would have to be able to evaluate $\nabla f_k(\mathbf{x})$ by some method, or least to obtain bounds for each component $\nabla^i f_k(\mathbf{x})$ that would allow the proof to go through.

The large increase in analytic complexity caused by the addition of even the simplest interaction term is cause for both consternation and challenge. The results presented here are meant to stimulate some of the challenge.

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