

## Improved Bounds for Aggregated Linear Programs

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A method of Kallio for improving bounds on the optimal value of a linear program calculated from an intermediate iteration is used to improve Zipkin's bounds for an aggregated linear program. Both theoretical and computational results are given, demonstrating the improvement due to these new bounds.

**O**NE METHOD for solving large scale linear programs (LPs) or large scale Markov decision processes (MDPs) is by first finding an exact solution to a smaller, aggregated problem; disaggregating this solution to the original problem; and finding bounds on the error from using the approximate solution. This has been studied extensively by Zipkin (1980) for fixed-weight aggregation.

In related work, Kallio (1977) derives bounds for a nonaggregated LP which is stopped at some iteration of the simplex method, and uses a decomposition technique and marginal analysis to tighten the bounds. For nonaggregated LPs, Kallio's Theorem 1 and Zipkin's Proposition 2 are identical. In this paper, Kallio's method is used to improve the bounds for aggregated LPs and for fixed-weight aggregated MDPs. The notation of this paper follows that of Zipkin whenever possible.

### 1. The Model

The original LP is

$$\begin{aligned} z^* = \max \quad & cx \\ \text{subject to} \quad & Ax \leq b \\ & x \geq 0 \end{aligned} \tag{1}$$

where  $c = (c_j)$  is an  $n$ -vector,  $b = (b_i)$  is an  $m$ -vector,  $A = (a_{ij})$  is an  $m \times n$  matrix, and  $x = (x_j)$  is an  $n$ -vector of variables.

Let  $\sigma = \{S_k: k = 1, \dots, K\}$  be an arbitrary partition of  $\{1, \dots, n\}$ , and

$n_k = |S_k|$ . Define  $\mathcal{A}^k$  to be the submatrix of  $A$  consisting of those columns whose indices are in  $S_k$ . Define  $c^k$  and  $x^k$  similarly. Let  $g^k$  be a non-negative  $n_k$ -vector whose components sum to unity, and define:

$$\bar{A}^k = \mathcal{A}^k g^k, \bar{c}_k = c^k g^k, k = 1, \dots, K.$$

Let  $\bar{A} = (\bar{A}^1, \dots, \bar{A}^K)$ ,  $\bar{c} = (\bar{c}_1, \dots, \bar{c}_K)$  and  $X$  a  $K$ -vector of variables. Then the weighted column aggregate problem of (1) is:

$$\begin{aligned} \bar{z} = \max \quad & \bar{c}X \\ \text{subject to} \quad & \bar{A}X \leq b \\ & X \geq 0. \end{aligned} \quad (2)$$

Let  $z^*$  be the optimal value of (1) and let  $x^*$ , an  $n$ -vector, be an optimal solution to (1). Zipkin shows that for any partition  $\sigma' = \{S_k': k = 1, \dots, K'\}$  of  $\{1, \dots, n\}$  such that there exist known positive numbers  $\{d_1, \dots, d_n\}$  and known non-negative numbers  $\{p_1, \dots, p_{K'}\}$  with:

$$\sum_{j \in S_k'} d_j x_j^* \leq p_k, k = 1, \dots, K' \quad (3)$$

then

$$\bar{z} \leq z^* \leq \bar{z} + \xi_a$$

where

$$\xi_a = \sum_{k=1}^{K'} [\max_{j \in S_k'} \{(c_j - \bar{u}A^j)/d_j\}]^+ p_k,$$

$\bar{u}$  is the vector of optimal dual variables of (2),  $A^j$  is the  $j$ th column of  $A$ , and  $(a)^+ = \max(0, a)$ .

Kallio assumes there are two known  $n$ -vectors,  $l$  and  $p$ ,  $0 \leq l \leq p \leq \infty$ , such that if the constraint  $l \leq x \leq p$  is added to (1), then the value of an optimal solution is unchanged. If  $\bar{z}$  is the value of a current feasible basis, and  $\bar{u}$  is the corresponding vector of simplex multipliers, Kallio's bounds are:

$$\bar{z} \leq z^* \leq \bar{u}b + \sum_j (c_j - \bar{u}A^j)^+ \cdot p_j + \sum_j (C_j - \bar{u}A^j)^- \cdot l_j \quad (4)$$

where  $(a)^- = \min(a, 0)$ .

Kallio shows how a certain restricted dual problem can be solved by marginal analysis to yield a tighter bound than does (4). In what follows, it is assumed  $d_j$  in (3) is identically 1 for all  $j$ . The extension to the more general case is straightforward. Extending this idea to the aggregated LP, amend to (1) the  $K$  constraints  $\sum_{j \in S_k} x_j \leq p_k, k = 1, \dots, K$ . Then the dual to this restricted problem is

$$\begin{aligned} \text{minimize} \quad & ub + \sum_{k=1}^K \delta_k p_k \\ \text{s.t.} \quad & \sum_i a_{ij} u_i + \delta_k \geq c_j, j \in S_k; k = 1, \dots, K, u \geq 0; \delta \geq 0. \end{aligned} \quad (5)$$

Let  $\bar{c}_j = c_j - \bar{u}\bar{A}'_j$ ,  $\bar{\delta}_k = \max_{j \in S_k} (\bar{c}_j)^+$  and  $\bar{\delta} = (\bar{\delta}_k)$ . By assumption, (1) and (5) have the same optimal value  $z^*$ .

**LEMMA 1.**  $(\bar{u}, \bar{\delta})$  is a feasible solution to (2.5).

Lemma 1 leads to the extension of Kallio's Theorem 1 for an aggregated LP. Consider (5) with  $u$  restricted to the set  $U = \{u | u = \theta\bar{u}, \theta \in R\}$ . The new restricted dual program is

$$\begin{aligned} & \text{minimize } \bar{z}\theta + \sum_{k=1}^K \delta_k p_k \\ & \text{s.t. } (\sum_i a_{ij}\bar{u}_i)\theta + \delta_k \geq c_j, \quad j \in S_k; j = 1, \dots, n, \\ & \quad \bar{u} \geq 0; \delta_k \geq 0, \quad k = 1, \dots, K \end{aligned} \quad (6)$$

Let  $z(\theta)$  be an optimal value of (6) as a function of  $\theta$ .

**THEOREM 1.**  $z(\theta)$  is a convex and piecewise linear function. Let  $z_-(\theta)$  be the left-hand derivative of  $z(\theta)$  with respect to  $\theta$ . Then the possible discontinuity points of  $z_-(\theta)$  where an optimum can occur are at  $\theta_1, \dots, \theta_n$  where:

$$\theta_j = \begin{cases} c_j / \sum_i a_{ij}\bar{u}_i & \text{if } \sum_i a_{ij}\bar{u}_i \neq 0 \\ \infty & \text{if } \sum_i a_{ij}\bar{u}_i = 0, \text{ for all } j. \end{cases}$$

*Proof.* The dual LP (6) can be decomposed into  $K$  subproblems, each of which has a solution value  $\max_{j \in S_k} (c_j - (\theta \sum_i a_{ij}\bar{u}_i))^+ p_k$ , which are convex and piecewise linear. This implies that  $z(\theta)$  is given by:

$$z(\theta) = \bar{z}\theta + \sum_{k=1}^K [\max_{j \in S_k} (c_j - (\theta \sum_i a_{ij}\bar{u}_i))]^+ p_k$$

which is convex and piecewise linear since it is the sum of such functions. The rest of the theorem follows from this property and inspection of the formula for  $z(\theta)$ .

Let  $z(\theta^*)$  be the minimum value of  $z(\theta)$ , that is an optimal solution to (6) over all possible values of  $\theta$ .

**COROLLARY 1.**  $\bar{z} \leq z^* \leq z(\theta^*) \leq \bar{z} + \xi_a$ .

*Proof.*  $z(\theta^*) \leq \bar{z} + \xi_a$  since  $\bar{z} + \xi_a$  is equivalent to  $z(1)$ , and  $\theta^*$  minimizes  $z(\theta)$ . By weak duality, a solution to (5) is greater than or equal to a solution of its dual problem. Since the constraint  $u \in U$  restricts (5),  $z(\theta^*)$  is no less than an optimal value to (5), hence  $z(\theta^*)$  is a legitimate upper bound.

Following and extending Kallio, for any value of  $\theta$ , define the index  $j(\theta, k)$  to be the  $j$  where  $(c_j - \theta \sum_i a_{ij}\bar{u}_i)$  is maximized for  $j \in S_k$ ,  $k = 1, \dots, K$ . Define the set  $J(\theta) = \{j | j = j(\theta, k), k = 1, \dots, K\}$ , and define the set  $I(\theta) = \{j | j \in J(\theta); \theta_j \leq \theta \text{ and } \sum_i a_{ij}\bar{u}_i < 0 \text{ or } \theta_j > \theta \text{ and } \sum_i a_{ij}\bar{u}_i \geq 0\}$ . Then  $z_-(\theta) = \bar{z} - \sum_{j \in I(\theta)} p_k (\sum_i a_{ij}\bar{u}_i)$ . Using this formula for  $z_-(\theta)$ , the marginal analysis to find  $z(\theta^*)$  proceeds as in Kallio. At  $\theta^*$ ,  $z(\theta^*)$  can be

evaluated by the formula  $z(\theta^*) = \bar{z}\theta^* + \sum_{j \in I(\theta^*)} (c_j - \theta_i^* a_{ij} \bar{u}_i) p_k$ . This involves evaluating at most  $K$  terms.

*Examples.* This example is from Zipkin. The original problem is:

$$\begin{aligned} z^* &= \max 2.5x_1 + 3x_2 + 4x_3 + 5x_4 \\ &\text{subject to } 4x_1 + 5x_2 + 7x_3 + 10x_4 \leq 54 \\ &\quad x_1 + 2x_2 + x_3 + 2x_4 \leq 10 \\ &\quad x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

An optimal solution is  $x_1^* = 16/3$ ,  $x_3^* = 14/3$ ,  $x_2^* = x_4^* = 0$ ,  $z^* = 32$ . Zipkin solves two aggregate problems. For the first,  $K = 2$ ,  $s_1 = \{1, 2\}$ ,  $s_2 = \{3, 4\}$ , the column weights in each partition are (0.5, 0.5). The second aggregate problem changes the column weights to (0.75, 0.25). For the first problem,  $\bar{z} = 28\%$ ,  $\bar{u} = (2^{1/48}, 2^{5/48})$ , and with  $p_1 = 10$ ,  $p_2 = 8$ , the bounds are  $28\% \leq z^* \leq 34^{1/24}$ . For finding the improved bound,  $\theta_1 = 1.101$ ,  $\theta_2 = 0.4290$ ;  $\theta_3 = 2.116$ ,  $\theta_4 = 0.9231$ . Then  $\theta^* = \theta_1 = 1.101$ , and the improved bound is  $28\% \leq z^* \leq 32.1855$ .

For the second aggregate problem, Zipkin's bounds are  $30\% \leq z^* \leq 33^{23/70}$ . For the improved bounds,  $\theta_1 = 1.051$ ,  $\theta_2 = 0.9817$ ;  $\theta_3 = 1.0606$ ,  $\theta_4 = 0.8794$ . At an optimum,  $\theta^* = \theta_1 = 1.051$ , and the improved bound is  $30\% \leq z^* \leq 32.1231$ . In both problems, the improved upper bound is an extremely tight bound on the true optimal value  $z^*$ .

In Mendelssohn (1978a-c) these results are extended to derive improved bounds by dominance for aggregated MDPs. A numerical example is presented for a real life model that has been suggested for use in managing salmon runs.

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#### REFERENCES

- KALLIO, M. 1977. Computing Bounds for the Optimal Value in Linear Programming. *Naval Res. Logist. Quart.* **24**, 301-308.
- MENDELSSOHN, R. 1978a. Bounds by Dominance for Fixed-Weight Aggregated Markov Decision Processes. SWFC Admin. Rep. 13H, 14 p.
- MENDELSSOHN, R. 1978b. Improved Bounds for Row Aggregated Linear Programs: A Note. SWFC Admin. Rep. 17H, 4 p.
- MENDELSSOHN, R. 1978c. Improved Bounds for Aggregated Linear Programs (unabridged version). SWFC Admin. Rep. 16H, 18 p.
- ZIPKIN, P. 1980. Bounds on the Effect of Aggregating Variables in Linear Programs. *Opns. Res.* **28**, 903-916.