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Notes, Comments, and Letters to the Editor

Capital Accumulation and the Optimization of Renewable Resource Models*

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1. INTRODUCTION

There are separate literatures on normative models of capital accumulation, fisheries management, and reservoir operation. However, generic models of each type share a common mathematical structure. The models often include uncertainty in the length of time to which planning should apply and in the future consequences of present decisions.

The present paper investigates the structure of optimal decision making under uncertainty for general single sector growth models, for individual optimal consumption and savings models, for models of a single fish species with pooled age classes, and for models of a single reservoir. The results extend and unify some of those of Amir (1967), Bewley (1977), Levhari and Srinvinasan (1969), Hakaanson (1970), Mirman (1971), Brock and Mirman (1972, 1973), Miller (1974), Sobel (1975), Whitt (1975a), Schechtman (1976), Schechtman and Escudero (1977), and Yaari (1976). Also we present new and simpler proofs for most of the theorems which generalize results of the authors above.

For brevity, we use the terminology of capital accumulation. Let the first consumption and reinvestment decisions be made in period n and the last

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ones in period 1. At the beginning of each period t ($t = 1, 2,\ldots$) let x_t denote the capital on hand in units of dollars or physical quantities as the context dictates. The decisions in period t are y_t, namely the amount of x_t that is reinvested, and $z_t = x_t - y_t$, the amount consumed. Let

$$y_t \in Y(x_t) \qquad (t = 1, 2,\ldots) \tag{1.1}$$

constrain the reinvestment and consumption decisions. The connection between reinvestment decisions and accumulated capital is

$$x_{t+1} = s(y_t, D_t), \tag{1.2}$$

where $D_1, D_2,\ldots, D_n$ are assumed to be independent random variables that are distributed as the generic random variable D. We assume for each t that x_t lies in a convex reference set X and that $C \equiv \{(x, y): y \in Y(x), x \in X\}$ is a convex set.

Let $G(x_t, y_t)$ denote the utility in period t of having an initial capital x_t and reinvesting y_t. Let α denote the single-period discount factor. The generic problem is maximization of expected discounted utility, namely $E\sum_{t=1}^{n} \alpha^{t-1}G(x_t, y_t)$, where $n \leqslant \infty$. With a *consumption horizon* of n periods until termination, let $A_n(x)$ denote an optimal reinvestment decision and $x - A_n(x)$ an optimal consumption decision. (For a fishery model, $x - A_n(x)$ is the amount harvested, and $A_n(x)$ is the population size after harvesting ceases. For a reservoir model, x is the amount of water in the reservoir, and $x - A_n(x)$ is the amount discharged.)

The real line is indicated by $\mathbf{R}$, and $\mathbf{R}_+$ denotes $[0, \infty)$. If $z \in \mathbf{R}$ then $(z)^+$ denotes $\max(0, z)$. Derivatives (partial or regular) are from the left when necessary. If $w(\cdot, \cdot)$ is a function of two variables, then $w^{(1)}(u, v)$ and $w^{(2)}(u, v)$ denote the partial derivatives with respect to the first and second arguments. If $a \in \mathbf{R}$ and $b \in \mathbf{R}$, then $a \wedge b$ denotes $\min(a, b)$.

Suppose w is a real-valued concave function on $\mathbf{R}$ and D is a random variable for which the expectations $r(z) \equiv Ew(z - D)$ and $Ew'(z - D)$ both exist. Then $r'(z) = Ew'(z - D)$ can be justified by the monotonicity in δ of $[w(z - D) - w(z - D - \delta)]/\delta$ (because w is concave) and the Dominated Convergence Theorem (Royden 1963).

2. New Features of the Model

The results in this paper relax the assumptions of the previously cited papers in five important ways. First, many of our results do not assume that single-period utility depends only on consumption, that is $G(x, y) \equiv g(x - y)$. In renewable resource models, the benefits of a harvest $x_t - y_t$ are offset by harvesting costs that typically depend separately on the stock size x_t and the amount harvested $x_t - y_t$.

Second, when the utility function does depend only on consumption, we assume neither $g'(0) = +\infty$ nor strict concavity of $g(\cdot)$. Neither assumption is salient for many renewable resource models. However, one or both of the assumptions is found in most of the cited references, most particularly Bewley (1977), Brock and Mirman (1972), Mirman and Zilcha (1975), and Schechtman (1976). Thus their models fail to encompass either linear utilities or quadratic utilities, or both. These cases are empirically useful.

Third, we avoid assumptions about $s(\cdot, \cdot)$ (in (1.2)) that would block applications to renewable resource models. The Inada condition

$$s^{[1]}(0, \cdot) \equiv +\infty, \qquad s^{[1]}(\infty, \cdot) \equiv 0$$

are not imposed. Note that the simple case $s(y, d) = \rho y + d$ violates the Inada condition as do quadratic functions. For several of the results it is not assumed that $s(\cdot, d)$ is nondecreasing for each fixed value d of D_t. We avoid continuity and ordering assumptions about $s(y, \cdot)$ such as are found in Brock and Mirman (1972), Schechtman (1976), and Bewley (1977). When $G(x, y) = p \cdot (x - y)$, new results are presented that assume only pseudoconcavity of $s(\cdot, d)$ for each fixed value d of D_t. This last assumption is often made in models of renewable resources.

Fourth, *effective constraints* are allowed in the model. We do not assume existence of interior solutions. In renewable resource models, the absence of Inada conditions typically causes some constraints to be active at an optimum. This makes the analysis more complex because it is no longar possible to assume that the derivative of an optimal value function equals $g'(x, x - A_n(x))$. The methods of proof, therefore, differ significantly from those in previously cited papers.

Finally, a unified treatment is presented for both the finite and infinite horizon problems. Our proof that the infinite horizon optimal policy $A(x)$ is the limit of the $A_n(x)$'s is more straightforward than those of Schechtman and Escudero (1977) (who assume $s(y_t, D_t) = \rho y_t + D_t$) and Brock and Mirman (1972) (who assume strict concavity, interior solutions and impose other restrictive conditions) although our model is sufficiently general to encompass renewable resource applications.

Also, we present a straightforward short proof that a unique stationary distribution of wealth exists. The proof uses a result of Rosenblatt (1967) and is much shorter and simpler than proofs of essentially the same theorem by Brock and Mirman (1972), Brock and Majumdar (1975), and Mirman and Zilcha (1975).

The model in Bewley (1977) is richer probabilistically than ours. Although his model requires $G(x, y) \equiv g(x - y)$ and $s(y, d) \equiv y + d$, g is stochastic with $\{(g_t, D_t)\}$ being a stationary stochastic process.

3. Assumptions

The following assumptions are made in various combinations.

(3.1) G is finite, concave, and continuous on C;

(3.2) for each y, $G(\cdot, y)$ is nondecreasing on $\{x: (x, y) \in C\}$;

(3.3) $G(x, y+\gamma) - G(x, y) \leqslant G(x+\lambda, y+\gamma) - G(x+\lambda, y)$ for all $\gamma > 0$, $\lambda > 0$ with all arguments in C;

(3.4) G is nonnegative and continuous on C;

(3.5) for each d, $s(\cdot, d)$ is continuous and concave on the set $\mathbf{Y} \equiv \bigcup_{x \in X} Y(x)$;

(3.6) for each d, $s(\cdot, d)$ is continuous and concave on the set $[0, y^o(d)]$ and convex on the set $[y^0(d), \infty)$;

(3.7) for each d, $s(\cdot, d)$ is nondecreasing on $\mathbf{Y}$;

(3.8) $G(x, y) = p \cdot (x - y), p \geqslant 0$;

(3.9) $Y(x) = [0, x], x \in X$.

Assumption (3.4) is slightly deceptive because several articles, including Phelps (1962), Hakaason (1970), and Miller (1974), have $G(x, y) = \log(x - y)$, for $x - y \geqslant \xi > 0$. However, nonnegativity of G is equivalent to having a uniform lower bound. Suppose, for example, that $G(x, y) \geqslant -B$ if $(x, y) \in B$ $(B > 0)$, and let $G^*(x, y) = G(x, y) + B$. Then a policy is optimal with utility function G^* if and only if it is optimal also for G. Moreover, G^* is nonnegative on C. Assumption (3.3) is equivalent to *supermodularity* of G on C. Topkis (1978) discusses supermodularity and its consequences for optimization.

4. Optimal Policies

From standard dynamic programming arguments, the generic problem of maximizing $E \sum_{t=1}^{\infty} \alpha^{t-1} G(x_t, y_t)$ leads to the recursion

$$f_n(x) = \sup\{J_n(x, y): y \in Y(x)\}, \qquad x \in X, \tag{4.1}$$

$$J_n(x, y) = G(x, y) + \alpha E f_{n-1}(s[y, D]), \qquad y \in Y(x), \quad x \in X \tag{4.2}$$

for $n = 1, 2, \ldots$ with $f_0(\cdot) \equiv 0$. The proofs of most results will exploit the following diminishing returns property of $f_n(\cdot)$.

Theorem 4.1. *Assumptions* (3.1), (3.2), (3.5), *and* (3.9) *imply for each n that $f_n(\cdot)$ is continuous, concave, and nondecreasing on X.*

Proof. The proof that $f_n(\cdot)$ inherits concavity and is nondecreasing is 'ell known and will not be reproved. Our proof of continuity is new and impler than related proofs in Brock and Mirman (1972) and Schechtman 1976).

Let p be a metric on the space in which X lies. To initiate a contrapositive roof, suppose f_n experiences a discontinuity at $x^0 \in X$. Then there is a $^0 \in Y(x^0)$ and a real number $\gamma > 0$ with the property that for all real umbers $\delta > 0$ there exists $x \in X$ such that

$$p(x, x^0) < \delta \qquad \text{and} \qquad J_n(x, y) \geqslant \gamma + J_n(x^o, y^o), \qquad y \in Y(x). \quad (4.3)$$

Concavity precludes the reverse inequality for J). Because C is a convex set is possible to select a subsequence (x^j, y^j) satisfying (4.3) and $(x^j, y^j) \to$ $x^0, y^0)$. Therefore, contrary to assumption, J_n is discontinuous at (x^0, y^0). ∎

Continuity of $J_n(x, \cdot)$ on $Y(x)$ for each $x \in X$ follows from (3.1), (3.5), nd continuity of $f_{n-1}(\cdot)$ via Theorem 4.1. Compactness of $Y(x)$ via (3.9), ierefore, implies attainment of the supremum in (4.1) for each $x \in X$ and xistence of an optimal policy.

The next two theorems describe the dependence of $A_n(x)$ on x. They lead n Corollary 4.2) to sufficient conditions for $0 \leqslant dA_n(x)/dx \leqslant 1$. Many uthors have proved at least one side of this inequality for special cases of ur model.

THEOREM 4.2. *Assumptions* (3.1)–(3.3), (3.5), *and* (3.9) *imply for each n* iat *there exists $A_n(\cdot)$ with the property*

$$A_n(x') \geqslant A_n(x) \qquad \text{if} \quad x' \geqslant x. \quad (4.4)$$

Proof. The theorem would be true if, for $\delta > 0$,

$$J_n(x', A_n(x)) - J_n(x', A_n(x) - \delta) \geqslant J_n(x, A_n(x)) - J_n(x, A_n(x) - \delta) \geqslant 0.$$

he right inequality is implied by the optimality of $A_n(x)$. The left inequality 'ould be implied by $J_n^{[2]}(\cdot, y)$ being nondecreasing on $\{x: (x, y) \in C\}$. From 1.2),

$$J_n^{[2]}(x, y) = G^{[2]}(x, y) + \alpha E\{f'_{n-1}(s[y, D])\, s^{[1]}(y, D)\}$$

ɔ (3.3) completes the proof. ∎

Theorem 4.2 requires neither $G(x, y) \equiv g(x - y)$ nor strict concavity. It ssumes concavity of $s(\cdot, d)$ but not monotonicity. Theorem 4.3 obtains a ırther result when $G(x, y) \equiv g(x - y)$ and $Y(x) = [0, x]$.

THEOREM 4.3. *Assumptions* (3.1)–(3.3), (3.5), (3.9), *and* $G(x,y) \equiv g(x-y)$ *imply for each* n *that there exists* $A_n(\cdot)$ *with the property*

$$0 \leqslant A_n(x') - A_n(x) \leqslant x' - x \qquad \text{if} \quad x \leqslant x' \tag{4.5}$$

Proof. The left inequality is (4.4). To prove the right one, let $z = x - y$ and rewrite (4.1) and (4.2) as

$$f_n(x) = \sup\{H_n(x,z): 0 \leqslant z \leqslant x\}, \qquad x \in X,$$
$$H_n(x,z) = g(z) + \alpha E f_{n-1}(s[x-z, D]).$$

The right side of (4.5) would be valid if

$$\begin{aligned} 0 &\leqslant H_n^{[2]}(x+\delta, z) - H_n^{[2]}(x,z) \\ &= \alpha E\{f'_{n-1}(s[x-z,D])\, s^{[1]}(x-z,D) \\ &\quad - f'_{n-1}(s[x+\delta-z,D])\, s^{[1]}(x+\delta-z,D)\} \end{aligned} \tag{4.6}$$

for $\delta > 0$ as will be shown.

Concavity of $s(\cdot, d)$ implies

$$s^{[1]}(x+\delta-z,d) \leqslant s^{[1]}(x-z,d).$$

If $s^{[1]}(x+\delta-z,d) \geqslant 0$ then concavity implies $s(x-z,d) \leqslant s(x+\delta-z,d)$ so $0 \geqslant f'(s[x+\delta-z,d]) \leqslant f'(s[x-z,d])$ and

$$f_{n-1}(s[x+\delta-z,d])\, s^{[1]}(x+\delta-z,d) \leqslant f_{n-1}(s[x-z,d])\, s^{[1]}(x-z,d). \tag{4.7}$$

If $s^{[1]}(x+\delta-z,d) < 0 \leqslant s^{[1]}(x-z,d)$ then (4.7) is trivial. If $0 > s^{[1]}(x-z,d) \geqslant s^{[1]}(x-z+\delta,d)$ then $s(x-z,d) \geqslant s(x+\delta-z,d)$ because $s(\cdot,d)$ is concave. Concavity and monotonicity of $f_{n-1}(\cdot)$ and these inequalities yield

$$0 \leqslant f'_{n-1}(s[x-z,d]) \leqslant f'_{n-1}(s[x+\delta-z,d]),$$
$$0 > s^{[1]}(x-z,d) \geqslant s^{[1]}(x+\delta-z,d)$$

and, therefore, (4.7), which proves (4.6). ∎

COROLLARY 4.1. *The assumptions of Theorem* 4.3 *and* $0 < A_n(x') < x'$ *for some* $x' > 0$ *imply* $0 < A_n(x)$ *for all* $x \geqslant x'$.

COROLLARY 4.2. *The assumptions of Theorem* 4.3 *imply*

$$0 \leqslant dA_n(x)/dx \leqslant 1, \qquad x \in X^0, \tag{4.8}$$

where X^0 *denotes the interior of* X.

Proof. From Theorem 4.4, $A_n(\cdot)$ is nondecreasing so its discontinuities, if any, are upward jumps. These jumps are precluded by $A_n(x) - x$ being nonincreasing so $A_n(\cdot)$ is continuous. Monotonicity of $A_n(\cdot)$ implies differentiability except, possibly, on a set of measure zero where one-sided derivatives exist, so (4.6) implies (4.8). ∎

Corollary 4.2 treats a more general problem than Schechtman (1976) does and its proof seems more straightforward.

An optimal policy, $A_n(x)$ can be described in further detail if $Y(x) = [0, x]$ and $G(x, y) = p \cdot (x - y)$ for $p \geqslant 0$. After substitution and rearrangement of terms, the optimization problem becomes:

$$\text{maximize } E\left\{px_1 - \alpha^{T-1}y_T + p\sum_{t=1}^{T}\alpha^{t-1}(\alpha s[y_t, D_t] - y_t)\right\}$$

subject to $0 \leqslant y_t \leqslant x_t$, $t = 1, \ldots, T$. The first term, px_1, is fixed. The second term, $-\alpha^{t-1}y_T$, has $y_T \equiv 0$ for an optimal policy (if $\alpha > 0$). Therefore, an equivalent problem, in the sense of having the same optimal policy for all $n > 1$, is given by the following recursion:

$$f_0(\cdot) \equiv 0,$$

$$f_n(x) = \sup\{J_n(y) : 0 \leqslant y \leqslant x\},$$

where $J_n(y) = G(y) + \alpha E f_{n-1}(s[y, D])$ and $G(y) = p \cdot (\alpha E s[y, D] - y)$.

Let x_n^0 denote a global maximum of $J_n(y)$. For $x_n \geqslant x_n^0$, it is optimal to consume $x_n - x_n^0$. If $J_n(\cdot)$ is pseudoconcave for all n, then it is straightforward to prove that an optimal policy is given by:

$$A_n(x) = x \wedge x_n^0.$$

What conditions ensure pseudoconcavity of $J_n(y)$? Corollary 4.3 is an immediate extension of Theorem 4.1.

COROLLARY 4.3. *Assumptions* (3.4), (3.5), (3.8), *and* (3.9), *imply*:

(i) *$J_n(y)$ is concave and continuous,*

(ii) $A_n(x) = x \wedge x_n^0$. ∎

Suppose

$$s(y, d) = d\phi(y), \qquad P\{D \geqslant 0\} = 1, \tag{4.9}$$

and

$\phi(\cdot)$ is pseudoconcave and continuous with mode at y_m.

When both (3.6) and (4.9) are valid, $y^0(\cdot) \equiv y^0 \geqslant y_m$.

THEOREM 4.4. *Assumptions* (3.6), (3.9) *and* (4.9) *imply*

$$A_n(x) = x \wedge x_n^0$$

is optimal.

Proof. Let $\mu = E(D)$. At $n=1$, $x_1^0 = \inf\{y: \phi'(y) \leqslant (\alpha\mu)^{-1}\} \leqslant y^m \leqslant y^0$. We use $f_0(\cdot) \equiv 0$. The inductive assumption is that $f_{n-1}(\cdot)$ is concave nondecreasing on X. In

$$J_n'(y) = p(\alpha\mu\phi'[y] - 1) + \alpha\phi'(y)\, E(Df_{n-1}'[D\phi(y)]),$$

the first term is nonpositive if $y \geqslant y^m \geqslant x_1^0$ and the second term is nonpositive if $y \geqslant y^m$ because then $\phi'(y) \leqslant 0$ (while $f'(\cdot) \geqslant 0$ and $P\{D \geqslant 0\} = 1$). Therefore, $x_n^0 \leqslant y^m$ so $A_n(x) = x \wedge x_n^0$ and $f_n(x) = J_n(x \wedge x_n^0)$. It follows that $f_n(\cdot)$ is concave nondecreasing if $J_n(\cdot)$ is concave on $[0, x_n^0]$, which is now verified.

$$J_n(y) = p(\alpha\mu\phi[y] - y) + \alpha E(f_{n-1}[D\phi(y)]),$$

whose first term is concave on $[0, x_n^0]$ because $\phi(\cdot)$ is concave on $[0, y^0]$ and $x_n^0 \leqslant y^m \leqslant y^0$. Concavity of the second term is implied by the inductive assumption, $\phi(y)$ being concave nondecreasing on $[0, x_n^0]$, and $P\{D \geqslant 0\} = 1$. ∎

Theorem 4.4 shows that, if $s(y, d) = d\phi(y)$, the shape of $\phi(\cdot)$ beyond its mode doesn't effect an optimal policy in any significant way because an optimal policy always returns the state to the concave part of the curve.

The properties assumed for $s(\cdot, \cdot)$ can be relaxed by making further assumptions about the distribution of D. A stochastic kernel $K(x, y)$ is TP_2 if, for all $x_1 < x_2$ and $y_1 < y_2$, $K(x_1, y_1)\, K(x_2, y_2) \geqslant K(x_2, y_1)\, K(x_1, y_2)$. TP_2 kernels include the exponential and range families of densities which contain the binomial, Poisson, gamma, and normal (with fixed variance) densities.

THEOREM 4.5. *Assumptions* (3.9), (3.8) *with* $p \geqslant 0$, $s(\cdot, d)$ *pseudoconcave and continuous for each* d, *and* D *with a continuous density function that is* TP_2 *implies*:

$$A_n(x) = x \wedge x_n^0.$$

Proof. This theorem and Theorem 4.1 have similar proofs except it must be shown (a) that a convex combination (expectation) of pseudoconcave functions using a random variable with a TP_2 density is again pseudoconcave, and (b) that a nondecreasing nonnegative pseudoconcave function of a pseudoconcave function is again pseudoconcave.

The first claim is Theorem 5.1 of Chapter 3 in Karlin (1968). The second claim is proven here for differentiable functions. From the definition of pseudoconcave functions, $f(\phi[\cdot])$ is pseudoconcave if

$$f'(\phi[y])\,\phi'(y)(y'-y)\leqslant 0 \qquad \text{implies} \quad f(y')\leqslant f(y)$$

First, $f'(\cdot)\geqslant 0$ so the only pertinent case is $\phi'(y)(y'-y)\leqslant 0$. The pseudoconcavity of $\phi(\cdot)$ implies $\phi(y')\leqslant\phi(y)$. However, $f(\cdot)$ is nondecreasing and pseudoconcave, so $f(\phi[y'])\leqslant f(\phi[y])$. ∎

5. Effects of the Consumption Horizon

This section investigates the impact of the consumption horizon on the structure of an optimal policy and on its valuation. The following result presents sufficient conditions for a longer consumption horizon to induce a higher valuation, greater accumulation, and higher incremental benefits per unit of added capital.

THEOREM 5.5. *For each n and $x\in X$:*

(a) *Assumption* (3.4) *implies*

$$f_n(x)\leqslant f_{n+1}(x); \tag{5.1}$$

(b) *Assumptions*[1] (3.2), (3.3), (3.4), (3.7) *and* (3.9) *imply*

$$A_n(x)\leqslant A_{n+1}(x); \tag{5.2}$$

$$J_n(x,y+\gamma)-J_n(x,y)\leqslant J_{n+1}(x,y+\gamma)-J_{n+1}(x,y), \quad \gamma>0 \quad \textit{if} \quad (x,y)\in C \quad \textit{and} \quad (x,y+\gamma)\in C; \tag{5.3}$$

$$f_n(x+\gamma)-f_n(x)\leqslant f_{n+1}(x+\gamma)-f_{n+1}(x) \qquad \textit{if} \quad x+\gamma\in X, \quad \gamma>0. \tag{5.4}$$

Proof. (a) $f_0(\cdot)\equiv 0$ initiates a straightforward inductive proof of (5.1) that uses (3.4).

(b) Observe that (5.3) is supermodularity (cf. Topkins (1978)) of $J_n(x,y)$ in (y,n) for each x, and (5.4) is supermodularity of $f_n(x)$ in (x,n). If $r(a,b)$ is supermodular in (a,b) and $M(\cdot)$ is nondecreasing then $r(a,m[b])$ also is supermodular in (a,b). Hence, *if* $f_{k-1}(x)$ is supermodular in (x,k) for all $k\leqslant n-1$ then (4.2) and (3.7) imply supermodularity of $J_k(x,y)$ in (y,k) for all $k\leqslant n$.

If $J_k(x,y)$ is supermodular in (x,y,k) for all $k\leqslant n$ and if C is a lattice

[1] Instead of (3.9) and convexity of X, it is sufficient to assume that $Y(x)$ is a compact lattice for each x, C and X are lattices, and $Y(x)$ is *ascending* on X.

then Theorem 6.2 in Topkis (1978) implies $A_{k-1}(x) \leqslant A_k(x)$ for all $k \leqslant n$. Assumption (3.9) and convexity of X imply that C is a lattice so it remains to establish (5.4).

Assumption (3.2) implies $f_1(x+y) - f_1(x) \geqslant 0$ (Theorem 1) so $f_0(\cdot) \equiv 0$ implies (5.4) is valid for $n = 0$. Inductively, if (5.4) is valid for all $n \leqslant k-1$ then (5.3) is valid for all $n \leqslant k$. Then Theorem 4.3 in Topkis (1978) implies (5.4) for all $n \leqslant k$. ∎

We are grateful to Donald M. Topkis of Bell Laboratories for suggesting this line of proof. Our earlier version of Theorem 5.1(b) contained superfluous assumptions including concavity of G.

The next result concerns the limiting behavior of f_n and A_n as $n \to \infty$. Suppose

$$0 < \alpha < 1; \tag{5.7}$$

$$G^{[1]}(x, y) < \infty, \qquad (x, y) \in C; \tag{5.8}$$

$$G(x, \cdot) \text{ is nonincreasing on } Y(x), \qquad x \in X; \tag{5.9}$$

let $r_1(x) \equiv x$ and $r_{n+1}(x) \equiv s(r_n(x), D_n)$; then $E \sum_{n=1}^{\infty} \alpha^{n-1} r_n(x) < \infty$. (5.10)

THEOREM 5.2. (a) *Assumptions* (3.1)–(3.5), (3.7), *and* (3.9) *imply for each* $x \in X$ *existence of*

$$A(x) = \lim_{n \to \infty} A_n(x). \tag{5.11}$$

If $G(x, y) \equiv g(x - y)$ *then*

$$0 \leqslant A(x') - A(x) \leqslant x' - x, \qquad x \leqslant x. \tag{5.12}$$

(b) *Assumptions* (3.1), (3.2), (3.4), (3.5), (3.7), (3.9), *and* (5.7)–(5.10) *imply for each* $x \in X$ *existence of*

$$f(x) \equiv \lim_{n \to \infty} f_n(x) \tag{5.13}$$

with $f(\cdot)$ *being concave and nondecreasing on* X.

Proof. (a) From (5.2) $A_n(x) \leqslant A_{n+1}(x) \leqslant x$ for every n and x. Hence, monotone convergence yields (5.11) and, via Theorem 4.3, it yields (5.12).

(b) Optimality of $A_n(x)$, (5.9), (3.7), and $f_{n-1}(\cdot)$ nondecreasing imply

$$\begin{aligned} f_n(x) &= G(x, A_n(x)) + \alpha E f_{n-1}(s[A_n(x), D_n]) \\ &\leqslant G(x, 0) + \alpha E f_{n-1}(s(x, D_n)) \\ &\leqslant \sum_{t=1}^{n} \alpha^{t-1} E G(r_t(x), 0). \end{aligned}$$

Concavity of G implies

$$G(u,v) \leqslant G(u^0,v^0) + [G^{[1]}(u^0,v^0), G^{[2]}(u^0,v^0)] \begin{pmatrix} u-u^0 \\ v-v^0 \end{pmatrix}$$

for all (u^0, v^0) and $(u, v) \in C$. However, $G^{(2)} \leqslant 0$ from (5.9) so

$$f_n(x) \leqslant \sum_{t=1}^{n} \alpha^{t-1} E[G(x,0) + G^{[1]}(x,0)(r_t(x) - x)]$$

$$\leqslant [G(x,0) - xG^{[1]}(x,0)]/[1-\alpha] + G^{[1]}(x,0) \sum_{t=1}^{\infty} \alpha^{t-1} r_t(x),$$

which is finite from (5.8) and (5.10). Therefore, $f_1(x), f_2(x), \ldots,$ is a bounded monotone sequence which implies (5.13). Monotone convergence implies that $f_1(\cdot), f_2(\cdot), \ldots,$ endow $f(\cdot)$ with their properties of concavity and monotonicity. ▮

The next result uses a familiar argument from inventory theory (Sobel 1970a) to prove that $f(\cdot)$ satisfies a functional equation analogous to (4.1). Let

$$J(x,y) = G(x,y) + \alpha E f(s[y,D]),$$

which exists by virtue of the following proof. The proof is much simpler than those of similar theorems in Brock and Mirman (1972) and Schechtman (1976).

THEOREM 5.3. *Assumptions* (3.1), (3.2), (3.4), (3.5), (3.7), (3.9), *and* (5.7)–(5.10) *imply*

$$\begin{aligned} f(x) &= \sup\{J(x,y) : y \in Y(x)\}, \qquad x \in X, \\ &= J(x, A[x]). \end{aligned} \tag{5.14}$$

Proof. For all x and n, $f_{n-1}(x) \leqslant f_n(x)$ so

$$J_{n-1}(x,y) \leqslant J_n(x,y) \leqslant f(x) \leqslant B(x) < \infty, \qquad y \in Y(x), \quad x \in X,$$

where $B(x)$ is the bound developed in the proof of (b) in Theorem 5.2. Therefore, $\{J_n(x,y)\}$ is a bounded monotone sequence so $J_n(x,y) \leqslant J(x,y)$ and

$$f_n(x) = \sup\{J_n(x,y) : y \in Y(x)\} \leqslant \sup\{J(x,y) : y \in Y(x)\}.$$

Convergence of f_n to f implies

$$f(x) \leqslant \sup\{J(x,y) : y \in Y(x)\}, \qquad x \in X,$$

so the theorem will have been proved after establishing

$$f(x) \geqslant \sup\{J(x, y): y \in Y(x)\}, \qquad x \in X. \tag{5.15}$$

Monotone convergence implies

$$f(x) \geqslant f_n(x) = \sup\{J_n(x, y): y \in Y(x)\},$$
$$f(x) \geqslant \lim_{n \to \infty} \sup\{J_n(x, y): y \in Y(x)\},$$

whereas the right side of (5.15) is

$$\sup\{\lim_{n \to \infty} J_n(x, y): y \in Y(x)\}.$$

Therefore, for (5.15) it is sufficient to prove

$$\lim_{n \to \infty} \sup\{J_n(x, y): y \in Y(x)\} = \sup\{\lim_{n \to \infty} J_n(x, y): y \in Y(x)\}. \tag{5.16}$$

The existence of the limit on the left side of (5.16) is implied by (5.13). For each n, $J_n(x, \cdot)$ is continuous on $Y(x) = [0, x]$ so $J_n(x, \cdot) \to J(x, \cdot)$ uniformly on $[0, x]$ because the Dominated Convergence Theorem implies $Ef_{n-1}(s[\cdot, D]) \to Ef(s[\cdot, D])$. Therefore,

$$0 = \lim_{n \to \infty} \sup\{J(x, y) - J_n(x, y): 0 \leqslant y \leqslant x\},$$

which implies (5.16) and, consequently, (5.15). ∎

$A(\cdot)$ inherits the properties of $\{A_n(\cdot)\}$.

COROLLARY 5.1. *The assumptions of Theorem* 5.2(a) *imply*

$$0 \leqslant A'(x) \leqslant 1. \tag{5.17}$$

6. ACCUMULATION USING A STATIONARY POLICY

Suppose the same policy $A(\cdot)$, arbitrary and possibly suboptimal, but satisfying (5.12), is used each period. Then reinvestment and consumption each period t are given by $A(\chi_t)$ and $\chi_t - A(\chi_t)$, where χ_t denotes the random asset level at the beginning of period t. Successive asset levels are connected by

$$\chi_{t+1} = s[A(\chi_t), D_t], \qquad t = 1, 2, \ldots, \tag{6.1}$$

or, equivalently, by a kernel $K(A(x), \Gamma)$ which is the probability of being in

$\Gamma \subseteq X$ if $A(x)$ is the action taken at state $x \in X$ (cf. Feller (1971)). More formally, consider the state space X with a Borel field of subsets B. Then $K(\cdot, \cdot)$ is a probability kernel if $K(x, \cdot)$ is a probability measure on B for each $x \in X$ and $K(\cdot, \Gamma)$ is a B-measurable function for each $\Gamma \in \beta$.

Whether or not the Markov processes χ_t converge to a stationary distribution is an important question. Convergence results in certain cases are given in Brock and Mirman (1972), Brock and Majumdar (1975), Mirman and Zilcha (1975), Schechtman (1976), and Schechtman and Escudero (1977). By comparison with these papers, the approach here relies on properties of the stochastic kernel $K(\cdot, \cdot)$. Therefore, the proofs are more direct and do not in any essential way depend on scalar properties of x_t and y_t (although the theorems are presented only for this case).

$K(\cdot, \cdot)$ induces the following operator T which takes bounded measurable functions h into bounded measurable functions:

$$(Th)(x) = \int K(x, dy)\, h(y). \tag{6.2}$$

There is a dual representation which takes probability measures Q into probability measures; namely, for each $\Gamma \subseteq X$,

$$(VT)(\Gamma) = \int Q(dx)\, K(x, \Gamma) \tag{6.3}$$

(Rosenblatt 1967). The operator T is equicontinuous if it maps continuous functions into continuous function. For the remainder of this section, suppose X is a compact set, $0 \in X$, and $s(\cdot, d)$ is continuous; note that (5.12) implies continuity of $A(\cdot)$. Therefore, it is straightforward to show that $K(\cdot, \cdot)$ induces an equicontinuous family of transformations. Consider the following three conditions:

(A) $K(x, I) > 0$ for all open intervals $I \subseteq X$.

(B) There exists a compact subset L of X, such that for each $x \in L$, $K(x, L) = 1$ and the operator T defined in (6.3) is irreducible on L in the sense of Rosenblatt (1967, p. 476).

(C) Neither (A) nor (B) holds.

Condition (A) is satisfied, for example, by models where $s(\cdot, \cdot)$ is linear, as in Schechtman and Escudero (1977), where $s(y, d) = ry + d$. Condition (B) is often proven as a preliminary result, as in Section 3 of Brock and Mirman (1972). Condition (C) is important when determining if a stationary distribution concentrates all its mass at a single point. The three conditions separate the question of the existence of a unique stationary distribution from the question of the existence of at least one, perhaps many, invariant measures on the set X, given some policy $A(x)$. Some of the literature confuses the two questions.

THEOREM 6.1. *If X is compact and $K(\cdot, \cdot)$ induces an equicontinuous family of transformatrions, then*:

(i) *There is at least one invariant measure on X.*

(ii) (A) *implies there is only one invariant measure on X and it is the unique positive stationary distribution on X.*

(iii) (B) *implies there is a unique invariant measure for each closed irreducible subset L of X, and this measure is the unique positive stationary distribution on L.*

(iv) *Suppose also, for some $\gamma > 0$, $P\{s(A[x], D) \geqslant x\} = 1$ for all $x \in [0, \gamma]$. Let $x^* = \sup\{x: x \in X\}$ and suppose $L = \{x: \gamma \leqslant x \leqslant x^*\}$ is irreducible. Then there is a unique stationary measure on X which has positive probability only on open intervals that intersect with L.*

(v) *If 0 is an absorbing state then* (C) *implies that there is a unique stationary distribution concentrated at 0 (so $P\{\liminf x_t = 0\} = 1$ and the process tends to gets arbitrarily close to 0).*

Proof. Part (ii) is Theorem 2 in Feller (1971, p. 272). Parts (i), (iii), (iv), and (v) are implied by Theorems 3 and 4 in Rosenblatt (1967) and Theorem 2.4 in Jamison (1964). ▮

Boylan (1977) proves theorems similar to (iv) and (v) although his approach is different. Two examples illustrate the power of the theorem. First, for the models in Schechtman (1976) and Schechtman and Escudero (1977), part (ii) of the theorem immediately implies convergence to a unique stationary distribution. Second, the model in Brock and Mirman (1972) has an equicontinuous kernel on a compact set so it necessarily possesses an invariant measure. This avoids the lengthy argument in Section 4 of Brock and Mirman's paper. Moreover, our results depend on irreducibility of an operator on a *subset* of the set of states so one can focus on states that "communicate" rather than on fixed points of growth functions. This permits simplified proofs of the results in Section 3 of Brock and Mirman's paper. Boylan (1977) gives such a simplified proof.

7. EXTENSIONS

The results thus far concern a model whose structure is stationary over time and in which growth does not occur. These restrictions can be relaxed.

Suppose, for example, that utility alters with age or time as well as with wealth. Let $G_k(x, y)$ denote the utility function appropriate to consumption at age $n-k$. Then all results in Section 4 and Theorem 5.1 remain valid if G is replaced by G_n in the assumptions in Section 3. As one example that generalizes Hakaanson (1973), let $G_k(\cdot, \cdot) \equiv 0$ if $k > 1$.

If the random variables $D_n, D_{n-1}, \ldots, D_2, D_1$ exhibit a dependence, let $T_k(D_n, \ldots, D_{n-k+1})$ denote a statistic of $D_n, \ldots, D_{n-k+1}$ that is sufficient for D_t. Let ϕ_k be a function which maps (T_k, D_{n-k}) into T_{k-1}. Such functions must exist, because the entire past history is a sufficient statistic. Then the results in Sections 4 and 5 through Theorem 5.1 remained unchanged if each $Y_n(x)$ is a convex set and $Y_n(x) \subseteq Y_n(x')$ if $x' \geqslant x$ for each n. If $Y_n(x)$ is appropriately convergent in n for each x, then generalization of Theorems 3.2, 5.3, and Corollary 5.1 can be obtained.

Exogenous price processes arise when there is a random sequence of prices $\rho_1, \rho_2, \rho_3, \ldots$, such that

$$G(x, y, \rho) = u(\rho[x-y])$$

and the constraint set is now $Y(x, p)$ for each fixed value p of ρ. If the assumptions in Section 3 are valid for each fixed value p of ρ, then the results in Sections 4 and 5 are true for each fixed p. For example, Theorem 5.1 becomes $f_{n-1}(x, p) \leqslant f_n(x, p)$, $A_{n-1}(x, p) \leqslant A_n(x, p)$, etc.

Undiscounted Utilities

The results in Section 4, Theorems 5.1 and 6.1 are true for all values of the discount factor $\alpha \geqslant 0$. If $\alpha \geqslant 1$ then generally $f_n(x) \to \infty$ as $n \to \infty$ so Theorems 5.2b and 5.3 are uninteresting. The validity of Theorem 5.2a is an intriguing issue when $\alpha = 1$.

The overtaking criterion (see Brock and Mirman (1973); Brock and Majumdar (1975) and their references to the work of Von Weizsacker, Gale, and Gale's students; Denardo and Rothblum (1979)) is currently in vogue when utilities are undiscounted. Consider, instead, the average gain criterion from the theory of Markov decision processes, namely,

$$\Gamma(Z \mid x) \equiv \varliminf_{T \to \infty} \sum_{j=1}^{T} G(x_j, y_j)/T \tag{7.2}$$

and search for a *gain-optimal* policy Z^* such that

$$\Gamma(Z^* \mid x) = \sup_Z \Gamma(Z \mid x) \qquad \text{for all} \quad x \in X.$$

It can be shown that a policy is gain optimal if it is overtaking optimal (using $\overline{\lim}$ in the definition). The converse is generally false so gain optimality is a weaker criterion than overtaking optimality.

Blackwell (1962), Derman (1962), and subsequent writers have explored the connection between the discounted criterion as $\alpha \uparrow 1$ and the gain criterion. In order to apply Derman (1962) as in Sobel (1970b), suppose X is a finite set and $Y(x)$ is a finite set for each $x \in X$. Then C is a finite set so the set Λ_0 of mappings from X to Y (i.e., x is mapped into $Y(x)$ for each $x \in X$) is finite. Let Λ denote the finite subset of Λ_0 that comprises mappings satisfying (5.12). Thus Λ is finite too.

If $\alpha < 1$, Theorem 5.2(a) asserts the existence of an optimal policy that is a member of Λ. Let $A(x, \alpha)$ denote the dependence on $\alpha < 1$ of such a policy and let $\alpha_1, \alpha_2, \ldots$ satisfy $\alpha_k < 1$ for all k and $\alpha_k \to 1$. Then $A(\cdot, \alpha_k)$ is an element of Λ, a finite set, so the sequence $A(\cdot, \alpha_1), A(\cdot, \alpha_2), \ldots$, must contain a subsequence all of whose members are the same element of Λ. Let $A(\cdot) \in \Lambda$ denote the policy corresponding to such a subsequence $\alpha_{n(1)}, \alpha_{n(2)}, \ldots$, i.e., $A(\cdot) \equiv A(\cdot, \alpha_{n(1)}) \equiv A(\cdot, \alpha_{n(2)}) \equiv \cdots$. Then the argument in Derman (1962) establishes

$$\Gamma(A(\cdot)|x) = \lim_{k \to \infty} (1 - \alpha_{n(k)}) f(x, \alpha_{n(k)}), \qquad x \in X,$$

$$\Gamma(A(\cdot)|x) = \sup_Z \Gamma(Z, x), \qquad x \in X,$$

where $f(x, \alpha)$ makes explicit the dependence of $f(\cdot)$ on the value of $\alpha < 1$. Therefore, $A(\cdot)$ is gain optimal. But $A(\cdot) \in \Lambda$ so $A(\cdot)$ satisfies (7.3).

Let $\mathbf{N}$ denote the set of natural numbers $\{0, 1, 2, \ldots\}$. Then the preceding argument justifies the following claim.

THEOREM 7.1. *If* $X = \{0, 1, 2, \ldots, \sigma\}$ *for some* $\sigma \in \mathbf{N}$ and $Y(x) = \{[0, x] \cap \mathbf{N}\}$, $x \in X$ *then the assumptions of Theorem* (5.2a) *imply existence of a gain-optimal policy that satisfies* (5.12).

It would be interesting to verify Theorem 7.1 without restricting X and $Y(x)$ for each x to be finite sets. If only X is finite, then the problem still seems surprisingly delicate. Suppose in this case there exists

$$A(x) \equiv \lim_{\alpha \uparrow 1} A(x, \alpha), \qquad x \in X. \tag{7.4}$$

Results of Fox (1967) show that $A(\cdot)$ may *not* be gain optimal unless the Markov chain structure induced by $A(\cdot)$ is also the chain structure induced by every one of $A(\cdot, \alpha_k)$, $k = 1, 2, \ldots$, where $\alpha_k \uparrow 1$.

The difficulty of establishing (7.4) is another obstacle to generalizing Theorem 7.1. The difficulty does not seem to stem from finiteness of X. One might conjecture that optimal consumption is a nonincreasing function of α so that $A(x, \cdot)$ is nondecreasing for each $x \in X$. Then the limit in (7.4) would exist because $A(x, \alpha) \in Y(x)$ so $A(x, \alpha) \leqslant x$ from $Y(x) = [0, x]$ and $A(x, \cdot)$

would be a bounded monotone function. To establish monotonicity of $A(x, \cdot)$ one might exploit Theorem 5.2(a) and first establish monotonicity of $A_n(x, \cdot)$ for all n (let the dependence on α of $A_n(x)$ and $f_n(x)$ be explicit). A straightforward inductive proof shows for each n that $f_n(x, \alpha)$ is concave and nondecreasing as a function of α. However, this property does not ensure monotonicity for $A_n(x, \cdot)$. An argument similar to the proof of Theorem 4.2 shows that a sufficient condition would be $J_n^{(3)}(x, y, \alpha)$ nondecreasing in α (for each n, x, and y). In turn, $J_n^{(2)}(x, y, \cdot)$ would be nondecreasing if $f_{n-1}^{(1)}(x, \alpha)$ were a nondecreasing function of α (for each x). This last step has not been accomplished nor is there a counter example.

Bewley (1977) proves, essentially, that $A(x, \cdot)$ is nondecreasing for each $x \in X$ when $X = \mathbf{R}_+$, $Y(x) = [0, x]$, $s(y, d) \equiv y + d$, and $G(x, y) \equiv g(x - y)$ is differentiable and strictly concave. The steps of his proof are similar to those above which we had previously outlined.

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